

On the sharp regularity for arbitrary actions of nilpotent groups on the interval: the case of N_4

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Abstract

In this work, we determine the largest α for which the nilpotent group of 4-by-4 triangular matrices with integer coefficients and 1 in the diagonal embeds into the group of $C^{1+\alpha}$ diffeomorphisms of the closed interval.

Introduction

This work deals with the next general two-fold question:

Given a group G of orientation-preserving homeomorphisms of a manifold M , is it conjugate to a group of diffeomorphisms of M ? If so, how smooth can this conjugate action be made?

In dimension larger than 1, the first half of the question has, in general, a negative answer, even for the action of a single homeomorphism [6]. However, in the case where M has dimension 1, this turns out to be very interesting, and the answer deeply depends on the dynamical/algebraic structure of the action/group considered. For instance, from the dynamical point of view, the classical Denjoy theorem says that a C^2 (more generally, C^{1+bv}) orientation-preserving circle diffeomorphism with irrational rotation number is necessarily conjugate to a rotation, hence minimal. On the other hand, in lower regularity, there are the so-called Denjoy counterexamples, namely, $C^{1+\alpha}$ diffeomorphisms with irrational rotation number that admit wandering intervals; besides, every circle homeomorphism is conjugate to a C^1 diffeomorphism. From the algebraic point of view, there is an important obstruction for a group G to admit a faithful action by C^1 -diffeomorphisms of a 1-manifold with boundary: every finitely-generated subgroup of G must admit a nontrivial homomorphism onto \mathbb{Z} (see [15]; see also [10] and [1]).

In this article, we focus on nilpotent group actions on the closed interval $[0, 1]$. (Extensions of our results to the case of the circle are left to the reader.) The picture for Abelian group actions is essentially completed by the works [3, 16]. For non-Abelian nilpotent groups, an important theorem of J.Plante and W.Thurston establishes that they do not embed in the group of C^2 -diffeomorphisms of $[0, 1]$ (see [13]). However, according to B.Farb and J.Franks, every finitely-generated, torsion-free nilpotent group can be realized as a group of C^1 diffeomorphisms of $[0, 1]$ (see also [7]). Motivated by this, we pursue the problem below, which was first addressed in [5] and stated this way in [9]. For the statement, recall that a diffeomorphism f is said to be of class $C^{1+\alpha}$ if its derivative is α -Holder continuous, that is, there exists $C > 0$ such that $|f'(x) - f'(y)| \leq C|x - y|^\alpha$ holds for all x, y .

Problem. *Given a nilpotent subgroup G of $\text{Homeo}_+([0, 1])$, find the largest α such that G embeds into the group $\text{Diff}_+^{1+\alpha}([0, 1])$ of $C^{1+\alpha}$ diffeomorphisms.*

There are two results in this direction. First, in [2] (see also [8]), the aforementioned Farb-Franks action of N_d , the nilpotent group of d -by- d lower triangular matrices with integer entries and 1 in the diagonal, is studied in detail. In particular, it is showed that this action cannot be made of class $C^{1+\alpha}$ for $\alpha \geq \frac{2}{(d-1)(d-2)}$, yet it can be made $C^{1+\alpha}$ for any smaller α . Second, a recent result of K.Parkhe [11] establishes that any action of a finitely-generated nilpotent group on $[0, 1]$ is topologically conjugate to an action by $C^{1+\alpha}$ -diffeomorphisms for any $\alpha < 1/\kappa$, where κ is the polynomial growth degree of the group.

For the particular case of N_4 , the regularity obtained by Parkhe is hence smaller than that of the Farb-Franks action, namely, $C^{1+\alpha}$ for $\alpha < 1/3$. Somehow surprisingly, even this regularity is not sharp, as it is shown by our

Theorem A. *The group N_4 embeds into $\text{Diff}_+^{1+\alpha}([0, 1])$ for every $\alpha < 1/2$.*

In [2], it is also shown that for any $d \in \mathbb{N}$, there is a nilpotent group of nilpotence degree d embedded into $\text{Diff}_+^{1+\alpha}([0, 1])$, for any $\alpha < 1$. (This is for instance the case of the Heisenberg group N_3 .) This suggests that the optimal regularity of a nilpotent group embedding into $\text{Diff}_+([0, 1])$ may not depend on the degree of nilpotence. Our second result shows that, at least, this invariant is not trivial, hence it is worth pursuing its study.

Theorem B. *The group N_4 does not embed into $\text{Diff}_+^{1+\alpha}([0, 1])$ for any $\alpha > 1/2$.*

We point out that the $C^{3/2}$ regularity is not covered by our results, though we strongly believe that N_4 does not admit an embedding in such regularity (compare [8]).

This article is organized as follows. In §1, we review some basic facts about the group N_4 such as normal forms. We also construct an action of N_4 on \mathbb{Z}^3 that preserves the lexicographic order on \mathbb{Z}^3 ; this action is inspired by the theory of left-orderable groups [4]. In §3, we show that for any $\alpha < 1/2$, the action of N_4 on \mathbb{Z}^3 can be projected into an action of N_4 on $[0, 1]$ by $C^{1+\alpha}$ diffeomorphisms, which shows Theorem A. Theorem B in turn is proved in §2.

All actions considered in this work are by orientation-preserving maps.

1 The group N_4

Throughout this work, we use the following notation. Given two group elements x, y , we let $[x, y] := xyx^{-1}y^{-1}$, and $x^y := yxy^{-1}$. Recall that the derived series of a group G is defined by $G^0 := G$ and $G^{i+1} := [G^i, G^i]$. The group G is solvable of degree d if G^d is trivial but G^{d-1} is not. The central series of G is defined by $G^{(0)} := G$ and $G^{(i+1)} := [G, G^{(i)}]$. The group G is nilpotent of degree ℓ if $G^{(\ell)}$ is trivial but $G^{(\ell-1)}$ is not.

The group N_4 is by definition the group of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ e & 1 & 0 & 0 \\ a & f & 1 & 0 \\ c & b & d & 1 \end{pmatrix}, \quad (1)$$

where all the entries belong to \mathbb{Z} . We will use the generating set S of N_4 consisting of the matrices for which all non-diagonal entries are 0 except for one which is 1. The elements of S will be denoted by e, f, d, a, b, c , where each of these elements represent the generating matrix

with a 1 in the position corresponding to the letter in (1); for example,

$$e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The reader can easily check that N_4 is isomorphic to the (inner) semidirect product $\langle f, a, b, c \rangle \rtimes \langle e, d \rangle$, where $\langle f, a, b, c \rangle \simeq \mathbb{Z}^4$ and $\langle d, e \rangle \simeq \mathbb{Z}^2$. The conjugacy action of \mathbb{Z}^2 on \mathbb{Z}^4 is given by

$$e : f \mapsto fa^{-1}, a \mapsto a, b \mapsto bc^{-1}, c \mapsto c, \quad (2)$$

$$d : f \mapsto fb, a \mapsto ac, b \mapsto b, c \mapsto c. \quad (3)$$

In particular, N_4 is metabelian (*i.e.* it has solvability degree 2). Further, N_4 has nilpotence degree 3: its lower central series is given by

$$N_4^{(1)} = \langle a, b, c \rangle, N_4^{(2)} = \langle c \rangle, N_4^{(3)} = \{id\}.$$

It follows from equations (2) and (3) that any element of N_4 can be written in a unique way as

$$f^{n_1} e^{n_2} d^{n_3} a^{n_4} b^{n_5} c^{n_6},$$

where the exponents n_i belong to \mathbb{Z} . This will be our preferred normal form. It allows proving the next

Lemma 1. *Let $\phi : N_4 \rightarrow G$ be a group homomorphism such that $\phi(c)$ is a nontrivial element of G with infinite order. Then ϕ is an embedding.*

Proof: We first observe that, for $(n_1, n_2) \neq (0, 0)$,

$$[\phi(d^{n_1} e^{n_2}), \phi(a^{n_1} b^{-n_2} c^{n_3})] = \phi([d^{n_1} e^{n_2}, a^{n_1} b^{-n_2} c^{n_3}]) = \phi(c^{n_1^2 + n_2^2}). \quad (4)$$

By the hypothesis, $\phi(c^{n_1^2 + n_2^2}) \neq id$, which implies that the restriction of ϕ to both $\langle a, b, c \rangle$ and $\langle d, e \rangle$ is an embedding.

Further, for $(n_1, n_2) \neq (0, 0)$, we have

$$\phi([d^{n_1} e^{n_2} a^{n_3} b^{n_4} c^{n_5}, a^{n_1} b^{-n_2}]) = \phi([d^{n_1} e^{n_2}, a^{n_1} b^{-n_2}]) = \phi(c^{n_1^2 + n_2^2}) \neq id,$$

thus the restriction of ϕ to $\langle d, e, a, b, c \rangle$ is an embedding. Finally we have that, for $n_0 \neq 0$,

$$\phi([f^{n_0} e^{n_1} d^{n_2} a^{n_3} b^{n_4} c^{n_5}, e]) = \phi(a^{n_0} c^{n_4}) \neq id.$$

Hence, ϕ is injective. □

Remark 1. An immediate consequence of Lemma 1 is that for every faithful action of N_4 by homeomorphisms of $[0, 1]$, there is a point $x_0 \in (0, 1)$ such that N_4 acts faithfully on its orbit. Indeed, it suffices to consider x_0 as being any point moved by c .

We next construct an action of N_4 by homeomorphisms of $[0, 1]$. Our method is close to the construction of Farb and Franks, who first built an action of N_4 on \mathbb{Z}^3 and then project it to an action on $[0, 1]$; see [5] or [2]. However, it should be emphasized that our action is different, which allows improving the regularity. We begin with

Proposition 1. Let $\tilde{e}, \tilde{f}, \tilde{d}, \tilde{a}, \tilde{b}$, and \tilde{c} be the maps from \mathbb{Z}^3 to \mathbb{Z}^3 defined by:

$$\begin{aligned}\tilde{e} &: (i, j, k) \mapsto (i+1, j, k), \\ \tilde{d} &: (i, j, k) \mapsto (i, j+1, k), \\ \tilde{f} &: (i, j, k) \mapsto (i, j, k-ij), \\ a' &: (i, j, k) \mapsto (i, j, k-j), \\ \tilde{b} &: (i, j, k) \mapsto (i, j, k+i), \\ \tilde{c} &: (i, j, k) \mapsto (i, j, k+1).\end{aligned}\tag{5}$$

Then the group \tilde{N} generated by $\langle \tilde{e}, \tilde{f}, \tilde{d}, \tilde{a}, \tilde{b}, \tilde{c} \rangle$ is isomorphic to N_4 .

Proof: It follows from the definition that $\tilde{f}, \tilde{a}, \tilde{b}$ and \tilde{c} commute, and that the subgroup of \tilde{N} that they generate is normal and isomorphic to \mathbb{Z}^4 . Further, the subgroup generated by $\{\tilde{e}, \tilde{d}\}$ is Abelian, and its action by conjugation on $\langle \tilde{f}, \tilde{a}, \tilde{b}, \tilde{c} \rangle$ mimics equations (2) and (3). Therefore, by Lemma 1, the application $x \mapsto \tilde{x}$, with $x \in \{e, d, f, a, b, c\}$, induces an isomorphism between N_4 and \tilde{N} . \square

We now let $(I_{i,j,k})_{(i,j,k) \in \mathbb{Z}^3}$ be a family of disjoint open intervals disposed on $[0, 1]$ respecting the (direct) lexicographic order of \mathbb{Z}^3 , that is, $I_{i,j,k}$ is to the left of $I_{i',j',k'}$ if and only if $(i, j, k) \prec (i', j', k')$, where \preceq is the lexicographic order on \mathbb{Z}^3 . Assume further that the union of this family of intervals is dense in $[0, 1]$. Then, by some abuse of notation, we can define e, d, f to be the unique homeomorphism of $[0, 1]$ whose restriction to each of the intervals $I_{i,j,k}$ is affine and send, respectively,

$$\begin{aligned}e &: I_{i,j,k} \mapsto I_{i+1,j,k}, \\ d &: I_{i,j,k} \mapsto I_{i,j+1,k}, \\ f &: I_{i,j,k} \mapsto I_{i,j,k+ij}.\end{aligned}\tag{6}$$

Since an affine map fixing a bounded interval must be the identity, Proposition 1 implies that the homeomorphisms e, d, f generate a subgroup of $\text{Homeo}_+([0, 1])$ isomorphic to N_4 . In order to show Theorem A, in §3, we will use, instead of affine maps, the so-called Pixton-Tsuboi family of local diffeomorphisms [14, 16].

Remark 2. At first glance, this action may look strange. However, it naturally appears when considering total order relations that are invariant under left-multiplication (*left-orders*, for short); see [4]. Namely, we may first endow the subgroups $\langle e, d \rangle \sim \mathbb{Z}^2$ and $\langle f, a, b, c \rangle \sim \mathbb{Z}^4$ with the left-orders \preceq_1 and \preceq_2 , respectively, and then consider the convex extension of these, which is a left-order \preceq on N_4 . In our construction, on the one hand, we let \preceq_1 be the lexicographic order for which e is (infinitely) larger than d . On the other hand, we let \preceq_2 be the lexicographic order in which c is the largest generator. Proceeding this way, the *dynamical realization* of the order \preceq is an action of N_4 on the real line that is semiconjugated into the action above.

2 Bounding the regularity

In this section, we show that the group N_4 does not embed in $\text{Diff}_+^{1+\alpha}([0, 1])$ provided that $\alpha > 1/2$. We first reduce Theorem B to a combinatorial statement, namely Lemma 2 below.

2.1 The combinatorics prevents an embedding

To state the main combinatorial lemma (whose proof is postponed to §2.2.), we introduce a notation that will be used throughout all §2.

Given a nilpotent group G acting by homeomorphisms of $[0, 1]$, a point $x_0 \in [0, 1]$, and an element $g \in G$, we define

$$J_g(x_0) := [\inf_n g^n(x_0), \sup_n g^n(x_0)].$$

Since G is nilpotent, given any $h \in G$, we have that the intervals $h(J_g)$ and J_g either are equal or have disjoint interior (otherwise, one can build a free subsemigroup inside G ; see for instance [4, §3.2]). In the latter case, we will say that h moves J_g . We have

Lemma 2. *Suppose that N_4 is faithfully acting on $[0, 1]$ by $C^{1+\alpha}$ -diffeomorphisms for some $\alpha > 1/2$. Then there exist g_1, g_2, g_3 in N_4 and $x_0 \in [0, 1]$ such that:*

1. $J_{g_3}(x_0)$ is not reduced to a point.
2. The element g_2 moves $J_{g_3}(x_0)$ and the element g_1 moves $J_{g_2}(x_0)$.
3. The elements g_1, g_2 , and g_3 pairwise commute. In particular, the subgroup $\langle g_1, g_2, g_3 \rangle$ is isomorphic to \mathbb{Z}^3 .

Lemma 2 provides us enough combinatorial information about the dynamics of N_4 to prove Theorem B. In concrete terms, looking for a contradiction suppose that N_4 is faithfully acting by $C^{1+\alpha}$ -diffeomorphisms for some $\alpha > 1/2$, and let g_1, g_2, g_3 and x_0 be the elements provided by the conclusion of Lemma 2. Then the only element in the Abelian group $\langle g_1, g_2, g_3 \rangle$ fixing x_0 is the trivial one. Further, by eventually changing some of g_1, g_2, g_3 by their inverses, we can suppose that they all move x_0 to the right. Hence, if we define $I_{0,0,0}$ as the interval $(x_0, g_3(x_0))$ and $I_{n_1, n_2, n_3} := g_1^{n_1} g_2^{n_2} g_3^{n_3} (I_{0,0,0})$, then the intervals $I_{i,j,k}$ are pairwise disjoint, they are disposed on $[0, 1]$ respecting the lexicographic order of the indices, and

$$g_1(I_{i,j,k}) = I_{i+1,j,k}, \quad g_2(I_{i,j,k}) = I_{i,j+1,k}, \quad g_3(I_{i,j,k}) = I_{i,j,k+1}.$$

A contradiction is then provided by the following theorem from [8] (see also [3])

Theorem 1. *Let $k \geq 2$ be an integer, and let f_1, \dots, f_k be commuting C^1 -diffeomorphisms of $[0, 1]$. Suppose that there exist disjoint open intervals I_{n_1, \dots, n_k} disposed on $(0, 1)$ respecting the lexicographic order and so that for all $(n_1, \dots, n_k) \in \mathbb{Z}^k$ and all $i \in \{1, \dots, k\}$,*

$$f_i(I_{n_1, \dots, n_i, \dots, n_k}) = I_{n_1, \dots, n_i+1, \dots, n_k}.$$

Then f_1, \dots, f_{k-1} cannot be all simultaneously of class $C^{1+1/(k-1)}$ provided that f_k is of class $C^{1+\alpha}$ for some $\alpha > 0$.

2.2 Proof of Lemma 2

As discussed in the previous section, in order to finish the proof of Theorem B, we need to prove Lemma 2. A first crucial step is given by the next result, which can be thought of as a version of Denjoy's theorem on the interval and corresponds to an extension of [3, Theorem C] for the case where the maps are not assumed to commute.

Theorem 2. *Given an integer $d \geq 2$ and $\alpha > 1/d$, suppose that G is a subgroup of $\text{Diff}_+^{1+\alpha}([0, 1])$ whose action is semiconjugated to a free action by translations of \mathbb{Z}^d . Then G acts minimally on $(0, 1)$, and it is hence Abelian.*

Proof: Looking for a contradiction, we suppose that the action of G is not minimal. We let I be a maximal open interval that is mapped into a single point by the semiconjugacy into a group of translations, and we let $f_1, \dots, f_d \in G$ be elements whose semiconjugate images generate \mathbb{Z}^d . Changing the f_i 's by their inverses if necessary, we may assume that they all move points inside $(0, 1)$ to the left.

We follow the proof of [9, Theorem 4.1.37], where the f_i 's are assumed to commute. Although in our situation the f_i 's do not *a priori* commute, they do commute on the closure Λ of the orbit of the endpoints of I . This allows applying all arguments of [9] except the last one, provided we consider the underlying Markov process directly on intervals. More precisely, assume that all the f_i 's are tangent to the identity at the origin (the other case works almost verbatim to [9]; alternatively, use the Müller-Tsuboi trick [16] to ensure flatness). Then consider the Markov process on \mathbb{N}_0^d with transition probabilities

$$p((n_1, \dots, n_i, \dots, n_d) \rightarrow (n_1, \dots, 1 + n_i, \dots, n_d)) := \frac{1 + n_i}{d + n_1 + \dots + n_d}.$$

Denote by Ω the space of infinite paths ω endowed with the induced probability measure \mathbb{P} . Let $S: \Omega \rightarrow \mathbb{R}$ be defined by

$$S(w) = \sum_{k \geq 0} |I_{\omega_k}|^\alpha,$$

where $w_k = (n_{1,k}, \dots, n_{d,k}) \in \mathbb{N}_0^d$ denotes the position of w at time k , and $I_{n_1, \dots, n_d} := f_1^{n_1} \dots f_d^{n_d}(I)$. Since $\alpha > 1/d$, this function has a finite expectation (see [3]). Thus, its value at a generic random sequence ω is finite. As in the proof of [9, Theorem 4.1.37], if for such a sequence we denote $h_k := f_1^{n_{1,k}} \dots f_d^{n_{d,k}}$, then we have

$$\frac{Dh_k(y)}{Dh_k(x)} \leq C \tag{7}$$

for all $k \geq 1$ and all x, y in $\bar{I} \cup \bar{J}$, where C only depends on ω and the α -Hölder constants of the derivatives of the f_i 's, and J is any interval that is next to I and has length smaller than $|I|/C$. By the maximality of I , there must exist some $h \in G$ mapping I into J . We then notice that, if J has endpoints in Λ , then for all $k \geq 1$ we have

$$\frac{|h(I)|}{|I|} = \frac{|h_k^{-1} h h_k(I)|}{|I|} = \frac{|h_k(I)|}{|I|} \cdot \frac{|h h_k(I)|}{|h_k(I)|} \cdot \frac{|h_k^{-1} h h_k(I)|}{|h h_k(I)|}.$$

In the product above, the middle quotient converges to $Dh(0) = 1$ as k goes to infinite. Besides, the first and the third quotients are respectively equal to $Dh_k(x_k)$ and $1/Dh_k(y_k)$ for certain points $x_k \in \bar{I}$ and $y_k \in \bar{J}$. Using (7), we conclude that $|h(I)|/|I| \geq 1/C$. However, this is impossible if J was chosen small-enough so that $|J| < 1/C$. \square

To finish the proof of Lemma 2, recall that every finitely-generated nilpotent group G of homeomorphisms of $(0, 1)$ preserves a nontrivial Radon measure μ on $(0, 1)$; see [12] or [9]. This measure induces a group homomorphism, the so-called *translation number homomorphism* $\tau_\mu: G \rightarrow \mathbb{R}$, whose kernel coincides with the set of elements in G having fixed points, and every such element must fix all points in $\text{supp}(\mu)$, the support of μ . Moreover, if $\tau_\mu(G)$ has rank greater than or equal to 2, then G is semiconjugate to the group of translations $\tau_\mu(G)$. In particular, from this we obtain

Lemma 3. *Suppose the Heisemberg group $N_3 \simeq \langle h_1, h_2, h_3 \mid [h_1, h_2] = h_3, h_i h_3 = h_3 h_i \ (i = 1, 2) \rangle$ is faithfully acting by homeomorphisms of $[0, 1]$. If x is not fixed by h_3 , then at least one of h_1, h_2 moves $J_{h_3}(x)$.*

We are now in position to give the

Proof of Lemma 2: Suppose N_4 faithfully acts by $C^{1+\alpha}$ diffeomorphisms of $[0, 1]$ for some $\alpha > 1/2$. We let x_0 be a point moved by c . By Remark 1, N_4 faithfully acts on its orbit. To simplify the notation, for $g \in N_4$, the interval $J_g(x_0)$ will be denoted by J_g .

The key observation is that in N_4 there are many isomorphic copies of the Heisemberg group N_3 so there are many instances in which we can apply Lemma 3. The reader can easily check that, for example, the subgroups

$$\langle e, b, c \rangle, \langle d, a, c \rangle, \langle f, d, b \rangle, \langle f, e, a \rangle$$

are all isomorphic to N_3 (the right-most generator being the generator of the center of N_3).

We let $g_3 := c$. Since J_c is not reduced to a point, the first part of the conclusion of Lemma 2 is satisfied. In order to find g_2 and g_3 , we distinguish two cases:

Case 1: Either a or b moves J_c .

Suppose a moves J_c . Then from Lemma 3 applied to $\langle f, e, a \rangle$ we have that either f or e moves J_a . Then we can let $g_2 := a$ and g_1 be an element in $\{f, e\}$ that moves J_a . For these elements the conclusion of Lemma 2 holds.

The case where b moves J_c works in the same way but looking at $\langle f, d, b \rangle$ instead of $\langle f, e, a \rangle$.

Case 2: Both a and b fix J_c .

Consider the group $\langle e, d \rangle \simeq \mathbb{Z}^2$ acting on the smallest possible interval containing x_0 , that is, the convex closure I of the $\langle e, d \rangle$ -orbit of x_0 . Observe that I is not contained in J_c since in that case both d and a would fix J_c , thus contradicting Lemma 3 applied to $\langle d, a, c \rangle$. In particular, the action of $\langle e, d \rangle$ on I is not minimal. Theorem 2 then implies that the action of $\langle e, d \rangle$ on I is not semi-conjugated to an action by translation of \mathbb{Z}^2 , so there must be $h_0 \in \langle e, d \rangle$ with translation number (over I) equal to zero.

If h_0 moves J_c we are done, since we can let $g_2 := h_0$ and g_3 be any element in $\langle e, d \rangle$ with non-trivial translation number. We claim that this is always the case; more precisely, we claim that any $h \in \langle e, d \rangle$ different from the identity moves J_c . Indeed, if $h = e^n d^m$ fixes J_c , then the group $H = \langle e^n d^m, a^m b^{-n} \rangle$ fixes J_c . But, if $(n, m) \neq (0, 0)$, then equation (9) implies that H is isomorphic to the Heisemberg group N_3 with center generated by $c^{n^2+m^2}$. A contradiction is then provided by Lemma 3 and the fact that $J_c = J_{c^k}$ for any $k \neq 0$.

This finishes the proof of the Lemma 2, and hence that of Theorem B. \square

3 The embedding

We next prove Theorem A. For the rest of this work, we fix α such that $0 < \alpha < 1/2$. In order to produce an embedding of N_4 into $\text{Diff}_+^{1+\alpha}([0, 1])$, we will project to the interval the action provided by Proposition 1 using the so-called Pixton-Tsuboi maps [14, 16]. This technique is summarized in the next

Lemma 4. *There exists a family of C^∞ diffeomorphisms $\varphi_{I', I}^{J', J} : I \rightarrow J$ between intervals I, J , where I' (resp. J') is an interval contiguous to I (resp. J) by the left, such that:*

1. (Equivariance) For all I, I', J, J', K, K' as above,

$$\varphi_{J',J}^{K',K} \circ \varphi_{I',I}^{J',J} = \varphi_{I',I}^{K',K}.$$

2. (Derivatives at the endpoints) For all I, I', J, J' ,

$$D\varphi_{I,I'}^{J,J'}(x_-) = \frac{|J'|}{|I'|}, \quad D\varphi_{I,I'}^{J,J'}(x_+) = \frac{|J|}{|I|},$$

where x_- (resp. x_+) is the left (resp. right) endpoint of I .

3. (Regularity) There is a constant M such that for all $x \in I$, we have

$$D \log(D\varphi_{I,I'}^{J,J'})(x) \leq \frac{M}{|I|} \cdot \left| \frac{|I|}{|J|} \frac{|J'|}{|I'|} - 1 \right|$$

provided that $\max\{|I'| |I|, |J'|, |J|\} \leq 2 \min\{|I'| |I|, |J'|, |J|\}$.

To produce our action, we let $I_{i,j,k}$ be a collection of intervals indexed by \mathbb{Z}^3 whose union is dense in $[0, 1]$ and that are disposed preserving the lexicographic order of \mathbb{Z}^3 . We then define the homeomorphisms d, e, f of $[0, 1]$ as those whose restrictions to $I_{i,j,k}$ coincide, respectively, with

$$\varphi_{I_{i,j,k-1}, I_{i,j,k}}^{I_{i+1,j,k-1}, I_{i+1,j,k}}, \quad \varphi_{I_{i,j,k-1}, I_{i,j,k}}^{I_{i,j+1,k-1}, I_{i,j+1,k}}, \quad \text{and} \quad \varphi_{I_{i,j,k-1}, I_{i,j,k}}^{I_{i,j,k+i-1}, I_{i,j,k+i}}.$$

By (Equivariance), this produces a faithful action of N_4 by homeomorphisms of $[0, 1]$.

Proposition 2. For an appropriate choice of the lengths $|I_{i,j,k}|$, the homeomorphisms e, f, d are simultaneously of class $C^{1+\alpha}$.

The rest of this work is devoted to the proof of this result. To begin with, we let p, q, r be positive reals for which the following conditions hold:

- (i) $\alpha + r \leq 2$,
- (ii) $4r \leq p$,
- (iii) $4r \leq q$,
- (iv) $4 \leq p(1 - \alpha)$,
- (v) $4 \leq q(1 - \alpha)$,
- (vi) $1/p + 1/q + 1/r < 1$,
- (vii) $\alpha \leq \frac{1}{p} + \frac{1}{r}$ and $\alpha \leq \frac{r}{p(r-1)}$,
- (viii) $\alpha \leq \frac{1}{q} + \frac{1}{r}$ and $\alpha \leq \frac{r}{q(r-1)}$.

For example, we can take $p = q := 4/\alpha$ and $r := 4/3$.

Now, let $I_{i,j,k}$ be an interval such that

$$|I_{i,j,k}| := \frac{1}{|i|^p + |j|^q + |k|^r + 1}.$$

Condition (vi) ensures that

$$\sum_{(i,j,k) \in \mathbb{Z}^3} |I_{i,j,k}| < \infty,$$

hence the $I_{i,j,k}$'s can be disposed on a finite interval respecting the lexicographic order. This interval can be thought of as $[0, 1]$ after renormalization.

Observe also that conditions (i) to (viii) can only be satisfied for $\alpha < 1/2$. Indeed, the second part of conditions (vii) and (viii) together imply that

$$2\alpha\left(1 - \frac{1}{r}\right) \leq \frac{1}{p} + \frac{1}{q}.$$

Then using (vi), one easily concludes that $2\alpha < 1$, that is, $\alpha < 1/2$.

It is proved in [2] that, with any choice of lengths as above, the maps e and d are $C^{1+\alpha}$ diffeomorphisms. More precisely, in [2, §3.3] it is shown that, under condition (vii), the diffeomorphism e is of class $C^{1+\alpha}$. Indeed, the second half of condition (vii) corresponds to condition (iii_B) in [2], while the first half of condition (vii), although not explicitly stated in [2], corresponds to the right form of condition (v_B) therein to show the regularity of e ; see, for instance, [2, page 125, line 9].

An analog argument applies to d under condition (viii). In order to conclude, below we will develop several slight modifications of some of the arguments of [2] to show the next lemma, which closes the proof of Theorem A.

Lemma 5. *For any choice of lengths of intervals satisfying properties (i),..., (viii) above, the homeomorphism f is a $C^{1+\alpha}$ diffeomorphism.*

Notice that this lemma is equivalent to that the expression

$$\frac{|\log Df(x) - \log Df(y)|}{|x - y|^\alpha}$$

is uniformly bounded (independently of x and y). To check this, due to property (*Derivatives at the endpoints*) above, it suffices to consider points x, y in intervals $I_{i,j,k}$ and $I_{i,j,k'}$, respectively; this means that the first “two levels” i and j coincide (compare [2, §3.3, III]). We will first deal with the case where the points x, y belong to the same interval $I_{i,j,k}$, and then with that where these points lie in intervals of this form but with different indices k, k' .

Case 1: The points x, y belong to the same interval $I := I_{i,j,k}$.

In this case, as $|x - y| \leq |I|$, from (*Regularity*) in Lemma 4 and the Mean Value Theorem we deduce that we need to find an upper bound for

$$\frac{1}{|I|^\alpha} \left| \frac{|I||J'|}{|I'||J|} - 1 \right|,$$

where J denotes $f(I) := I_{i,j,k+ij}$, $I' := I_{i,j,k-1}$, and $J' := I_{i,j,k+ij-1}$.

Case 2: The points x, y lie in different intervals, say $x \in I_{i,j,k}$ and $y \in I_{i,j,k'}$, with $k' > k$.

Here, using [16] (more precisely, [2, (20)]), it readily follows from the triangular inequality that $|\log Df(x) - \log Df(y)|$ is smaller than or equal to

$$\left| \log \frac{|I_{i,j,k+ij}|}{|I_{i,j,k}|} - \log \frac{|I_{i,j,k'+ij}|}{|I_{i,j,k'}|} \right| + \left| \log \frac{|I_{i,j,k+ij-1}|}{|I_{i,j,k-1}|} - \log \frac{|I_{i,j,k+ij}|}{|I_{i,j,k}|} \right| + \left| \log \frac{|I_{i,j,k'+ij-1}|}{|I_{i,j,k'-1}|} - \log \frac{|I_{i,j,k'+ij}|}{|I_{i,j,k'}|} \right|.$$

The last two terms in this sum are easy to estimate, as the indices k, k' do not mix in none of these. Hence, we need to estimate the first term. More precisely, we need to find an upper bound for

$$\frac{1}{|x - y|^\alpha} \left| \log \frac{|I||J'|}{|I'||J|} \right|,$$

where $I := I_{i,j,k}$, $J := f(I) = I_{i,j,k+ij}$ and $I' := I_{i,j,k'}$, $J' := f(I') = I_{i,j,k'+ij}$.

To deal with Cases 1 and 2 along the lines explained above, we introduce some notation. We say that two real-valued functions f, g satisfy $f \prec g$ if there is a constant M such that $|f(x)| \leq Mg(x)$ holds for all x . Observe that with this notation, for every $a > 0$, one has $(x+y)^a \prec \max\{|x|^a, |y|^a\}$. When f and g are non-negative functions, we will write $f \asymp g$ whenever $f \prec g$ and $g \prec f$. For instance, for $a > 0$, one has $|x+y|^a \asymp \max\{|x|^a, |y|^a\}$.

We would like to consider the family of functions $k \mapsto 1 + |i|^p + |j|^q + |k|^r$ together with their second derivatives. However, by (i) we have $r < 2$, hence these functions fail to be twice differentiable. This is why we instead consider the function

$$\varphi(i, j, \xi) := 1 + |i|^p + |j|^q + \theta(\xi),$$

where θ is a fixed C^2 function satisfying $\theta(\xi) = |\xi|^r$ for $|\xi| \geq 1$, and $\theta(0) = 0$. We then define the family of functions

$$G_{i,j}(\xi) := \log(\varphi(i, j, \xi)).$$

The following inequality will be of great importance for us: Let a_1, a_2, a_3 and b be non-negative real numbers such that $a_1/p + a_2/q + a_3/r \leq b$. Then since $|i| \leq \varphi^{1/p}(i, j, k)$, $|j| \leq \varphi^{1/q}(i, j, k)$ and $k \leq \varphi^{1/r}(i, j, k)$ hold for all integers i, j, k , we have

$$|i|^{a_1} |j|^{a_2} |k|^{a_3} \prec \varphi(i, j, k)^b. \quad (8)$$

We also have the following useful

Lemma 6. *Let $S := 1 + |i|^p + |j|^q$, and suppose $|\xi - k| \leq S^{1/r} + 2|ij|$. Then¹*

$$\varphi(i, j, \xi) \asymp \varphi(i, j, k).$$

Proof: By symmetry, it is enough to find a uniform bound for $\frac{\varphi(i, j, \xi)}{\varphi(i, j, k)}$, and this follows from

$$\frac{\varphi(i, j, \xi)}{\varphi(i, j, k)} = \frac{\varphi(i, j, k + (\xi - k))}{\varphi(i, j, k)} \prec 1 + \frac{|\xi - k|^r}{\varphi(i, j, k)} \prec 1 + \frac{S + 2^r |ij|^r}{\varphi(i, j, k)} \leq 2 + \frac{2^r |ij|^r}{\varphi(i, j, k)}$$

and the last expression is bounded due to conditions (ii) and (iii). \square

Now, consider the expression

$$\log \frac{|I||J'|}{|I'||J|} = \log |I| + \log |J'| - \log |I'| - \log |J|.$$

This can be seen as a “second increment” of the function $G_{i,j}$. Indeed, it equals

$$G_{i,j}(k + a + b) - G_{i,j}(k + a) - G_{i,j}(k + b) + G_{i,j}(k),$$

where, in case 1, $a = -1$ and $b = ij$, and in case 2, $a = k' - k$ and $b = ij$. An application of the Mean Value Theorem then yields

$$G_{i,j}(k + a + b) - G_{i,j}(k + a) - G_{i,j}(k + b) + G_{i,j}(k) = abG''_{i,j}(\xi) \quad (9)$$

¹Please notice that here (and also below) we are slightly abusing of the notation \asymp . Indeed, the precise conclusion should be that there is a universal constant M such that $\frac{1}{M}\varphi(i, j, \xi) \leq \varphi(i, j, k) \leq M\varphi(i, j, k)$ holds whenever $|\xi - k| \leq S^{1/r} + 2|ij|$.

where ξ is a certain point in $\text{conv}\{k, k+a, k+b, k+a+b\}$, the convex hull of $k, k+a, k+b, k+a+b$.

Now, for $\xi \notin [-1, 1]$, we have that

$$G'_{i,j}(\xi) = \frac{\varphi'}{\varphi} = \pm \frac{r\xi^{r-1}}{\varphi(i,j,\xi)} \prec \frac{\xi^{r-1}}{\varphi(i,j,\xi)} \quad (10)$$

and

$$G''_{i,j}(\xi) = \frac{\varphi''}{\varphi} - \left(\frac{\varphi'}{\varphi}\right)^2 = \pm \frac{r(r-1)\xi^{r-2}}{\varphi(i,j,\xi)} - \frac{r^2\xi^{2r-2}}{\varphi(i,j,\xi)^2} \prec \frac{\xi^{r-2}}{\varphi(i,j,\xi)}, \quad (11)$$

where the last bound holds since $\frac{r^2\xi^{2r-2}}{\varphi(i,j,\xi)^2} = \frac{r^2\xi^r}{\varphi(i,j,\xi)} \frac{\xi^{r-2}}{\varphi(i,j,\xi)}$, and the first factor of this product is always smaller than r^2 . Besides, for $\xi \in [-1, 1]$, the numerators of the right-side expressions in (10) and (11) are bounded from above by some constant independent of i, j . Therefore, for a general ξ , we have $G''_{i,j}(\xi) \prec \frac{1}{\varphi(i,j,\xi)}$.

Next, consider Case 1, that is, assume that x and y belong to the same interval. By (9) and (11), we have

$$G_{i,j}(k+ij-1) - G_{i,j}(k-1) - G_{i,j}(k+ij) + G_{i,j}(k) \prec |i||j| \frac{1}{\varphi(i,j,\xi)},$$

where ξ is certain point in $\text{conv}\{k, k-1, k+ij, k+ij-1\}$. But conditions (iv) and (v) imply that $|i||j| \prec \varphi^{1-\alpha}(i, j, k)$, and since changing k by $k \pm 1$ does not change the asymptotic behavior of $\varphi(i, j, k)$, from Lemma 6 we have

$$\log \frac{|I||J'|}{|I'||J|} \prec |i||j| \frac{1}{\varphi(i,j,\xi)} \prec \varphi^{-\alpha}(i, j, k).$$

In particular, except for finitely many indices $(i, j, k) \in \mathbb{Z}^3$, the value of $\varphi(i, j, k)^\alpha \log \frac{|I||J'|}{|I'||J|}$ is uniformly small. Therefore,

$$\frac{1}{|I|^\alpha} \left| \frac{|I||J'|}{|I'||J|} - 1 \right| \prec \frac{1}{|I|^\alpha} \log \frac{|I||J'|}{|I'||J|} \prec 1$$

holds for all indices, as desired.

Now consider Case 2, namely when x and y belong to different intervals $I := I_{i,j,k}$ and $I' := I_{i,j,k'}$, respectively. In this case, by (9) and (11), we have

$$G_{i,j}(k'+ij) - G_{i,j}(k') - G_{i,j}(k+ij) + G_{i,j}(k) \prec |ij(k'-k)| \frac{|\xi^{r-2}|}{\varphi(i,j,\xi)}, \quad (12)$$

where ξ is a certain point in $\text{conv}\{k, k', k+ij, k'+ij\}$. For simplicity, we will assume that $k' - k \geq 2$: the case $k' = k + 1$ follows from the previous one using property (*Derivatives at the endpoints*) just comparing at the right endpoints of $I_{i,j,k}$. Further, we also assume k, k' to be positive (the case where both are negative follows by symmetry, and if they have different sign, it suffices to consider an intermediate comparison with the term corresponding to $k'' = 0$). Finally, by eventually using the triangular inequality, we can restrict ourselves to three different regimes, namely when both k and k' belong to $[0, 2|ij|]$, or to $[2|ij|+1, S^{1/r}]$, or to $[S^{1/r}+1, \infty)$, where, as in Lemma 6, we denote $S := 1 + |i|^p + |j|^q$. In the same way, we can assume that x is the left end-point of I and y is the right end-point of I' .

Observe that the division into three intervals above is rather natural. The magnitude $S^{1/r}$ marks the point after which the size of the interval $I_{i,j,k}$ depends mainly on k and is comparable to $\frac{1}{k^r}$. This will be important to estimate the magnitude $|x - y|$ when k and k' are very far apart.

We start by noticing that in general

$$\frac{1}{|x - y|^\alpha} = \left(\frac{1}{\sum_{\ell=k}^{k'} |I_{j,k,\ell}|} \right)^\alpha \leq \left(\frac{1}{|k' - k| |I_{i,j,k'}|} \right)^\alpha.$$

Besides, if k, k' are such that $k' - k \leq S^{1/r}$, then $|I|$ and $|I'|$ are comparable by Lemma 6, hence

$$\frac{1}{|x - y|^\alpha} \prec \left(\frac{1}{|k' - k| |I_{i,j,k}|} \right)^\alpha. \quad (13)$$

We next separately deal with the three regimes $[0, 2|ij|]$, $[2|ij| + 1, S^{1/r}]$, and $[S^{1/r} + 1, \infty)$.

- Assume that k and k' lie in the interval $[0, 2|ij|]$. By Lemma 6, we have

$$\frac{1}{\varphi(i, j, \xi)} \asymp \frac{1}{\varphi(i, j, k)}.$$

Moreover, from conditions (iv) and (v) we deduce that $|ij||k' - k| \leq 2|ij|^2 \prec \varphi(i, j, k)^{1-\alpha}$. Also, as $r < 2$, we have $|\xi^{r-2}| \leq 1$. Therefore, by (12),

$$\frac{1}{|x - y|^\alpha} \left| \log \frac{|I||J'|}{|I'||J|} \right| \prec \frac{1}{|x - y|^\alpha} \frac{|ij||k' - k|}{\varphi(i, j, k)} \prec \frac{\varphi^{-\alpha}(i, j, k)}{|x - y|^\alpha} \prec \frac{\varphi^{-\alpha}(i, j, k)}{|I|^\alpha} = 1,$$

as desired.

- Assume that k and k' lie in the interval $[2|ij| + 1, S^{1/r}]$. Then, by (12) and (13), we need to obtain an upper bound for

$$\left(\frac{1}{|k' - k| |I_{i,j,k}|} \right)^\alpha |ij|(k' - k) \frac{|\xi^{r-2}|}{\varphi(i, j, \xi)}.$$

Since $S^{1/r} + |ij| \geq \xi \geq |ij|$ and $r < 2$, using Lemma 6 this reduces to estimating the expression

$$\left(\frac{1}{|I_{i,j,k}|} \right)^\alpha |ij|(k' - k)^{1-\alpha} \frac{|ij|^{r-2}}{\varphi(i, j, k)} = |k' - k|^{1-\alpha} \frac{|ij|^{r-1}}{\varphi(i, j, k)^{1-\alpha}}.$$

In other words, it is enough to show that $(k' - k)^{1-\alpha} |ij|^{r-1} \prec \varphi(i, j, k)^{1-\alpha}$. But since $k' - k \leq S^{1/r}$, this reduces to showing that

$$\left(|i|^{p(1-\alpha)/r} + |j|^{q(1-\alpha)/r} \right) |ij|^{r-1} \prec \varphi(i, j, k)^{1-\alpha}. \quad (14)$$

To show this, we claim that

$$|i|^{p(1-\alpha)/r} |ij|^{r-1} \prec \varphi(i, j, k)^{1-\alpha}$$

follows from (8). (The same inequality changing $|i|^{p(1-\alpha)/r}$ by $|j|^{q(1-\alpha)/r}$ follows in analogous way.) Indeed, in order to apply (8) we need to check that

$$\frac{1-\alpha}{r} + \frac{r-1}{p} + \frac{r-1}{q} \leq 1 - \alpha.$$

However, by (vi), we have $1/p + 1/q \leq 1 - \frac{1}{r}$. Therefore, it suffices to show that

$$\frac{1-\alpha}{r} + (r-1)\left(1 - \frac{1}{r}\right) \leq 1 - \alpha,$$

that is, $\alpha + r \leq 2$, which is nothing but condition (i).

- Finally, assume that k and k' are in the interval $[S^{1/r}, \infty]$. If $k' \leq 2k$, then

$$\frac{\varphi(i, j, k')}{\varphi(i, j, k)} = \frac{\varphi(i, j, k + (k' - k))}{\varphi(i, j, k)} \prec 1 + \frac{|k' - k|^r}{\varphi(i, j, k)} \leq 1 + \frac{|k|^r}{\varphi(i, j, k)} \leq 2.$$

Therefore, (13) still applies, so that we may proceed as in the second regime case above. One then easily checks that, instead of (14), now one needs to show that

$$|k|^{1-\alpha} |ij|^{r-1} \prec \varphi(i, j, k)^{1-\alpha},$$

which still holds thanks to (8) as above.

Assume now that $k' \geq 2k$. The key point in this case is that

$$\begin{aligned} |x - y| &= \sum_{\ell=k}^{k'} |I_{i,j,\ell}| = \sum_{\ell=k}^{k'} \frac{1}{1 + |i|^p + |j|^q + |\ell|^r} \\ &\prec \sum_{\ell=k}^{k'} \frac{1}{|\ell|^r} \\ &\prec \int_k^{k'} \frac{dx}{x^r} \\ &= \frac{1}{r-1} \left(\frac{1}{k^{r-1}} - \frac{1}{k'^{r-1}} \right) \\ &\geq \frac{1}{r-1} \left(1 - \frac{1}{2^{r-1}} \right) \frac{1}{k^{r-1}}. \end{aligned}$$

Thus, if we further estimate both $\log |I| - \log |J|$ and $\log |I'| - \log |J'|$ using the Mean Value Theorem, then we obtain that

$$\frac{1}{|x - y|^\alpha} \left| \log \frac{|I||J'|}{|I'||J|} \right| \prec k^{\alpha(r-1)} |ij| \left(\frac{\xi^{r-1}}{\varphi(i, j, \xi)} - \frac{\tilde{\xi}^{r-1}}{\varphi(i, j, \tilde{\xi})} \right)$$

for some points $\xi \in \text{conv}\{k, k + ij\}$ and $\tilde{\xi} \in \text{conv}\{k', k' + ij\}$. Since $\xi \mapsto \frac{\xi}{\varphi(i, j, \xi)}$ is a decreasing function, it suffices to obtain an upper bound for

$$k^{\alpha(r-1)} |ij| \frac{k^{r-1}}{\varphi(i, j, k)}.$$

In other words, we need to show that $k^{\alpha(r-1)} |ij| k^{r-1} \prec \varphi(i, j, k)$, which, by (8), follows provided we check that

$$\frac{(\alpha + 1)(r - 1)}{r} + 1/p + 1/q \leq 1.$$

But due to (vi), this holds whenever

$$\frac{(\alpha + 1)(r - 1)}{r} \leq \frac{1}{r},$$

that is, $(\alpha + 1)(r - 1) \leq 1$, or equivalently, $\alpha r + r \leq \alpha + 2$. However, using (i), we obtain

$$\alpha r + r = \alpha(r - 1) + (\alpha + r) \leq \alpha + 2,$$

as desired.

This finishes the proof of Theorem A.

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